Similarity rules for isothermal bubble growth

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A normalized formulation of the problem of isothermal bubble growth, dominated by viscosity and diffusion, reveals that in its full generality the problem involves eight dimensionless parameters. Limiting cases for extreme values of each of these parameters are investigated.

1. Introduction

In a previous paper, Barlow & Langlois (1962) considered the diffusion-fed growth of a spherical gas bubble into a Newtonian viscous liquid under isothermal conditions. A solute gas diffuses through the liquid according to Fick's law of diffusion for a medium in motion

$$\frac{\partial c}{\partial t}+v\frac{\partial c}{\partial r} = D\left(\frac{\partial^2 c}{\partial r^2}+\frac{2}{r}\frac{\partial c}{\partial r}\right). \tag{1.1}$$

In this equation, c denotes the gas concentration, D the diffusivity, r the distance from the bubble centre, t the time, and v the radial velocity of the liquid. The hydrodynamic equation of continuity requires that

$$v = R^2\dot{R}/r^2, \tag{1.2}$$

where R is the instantaneous radius of the bubble and an overdot denotes ordinary differentiation with respect to time.

It is assumed that, at any given time, conditions inside the bubble are homogeneous. The partial pressure p_g of solute gas is related to the concentration c_g according to Boyle's law

$$p_g = Ac_g, \tag{1.3}$$

where A is a constant. In general, there is also a residue pressure $p_r R_0^3/R^3$, arising from gases, assumed insoluble in the liquid, which may be trapped in the bubble at an initial partial pressure p_r. In the previous paper, the residue pressure was set equal to zero.

The rate of change of mass of solute gas inside the bubble is equal to the diffusion rate across the bubble wall, so that

$$d(\tfrac{4}{3}\pi R^3 c_g)/dt = 4\pi R^2 D(\partial c/\partial r)_{r=R}. \tag{1.4}$$

Initially, a uniform concentration c_0 of solute gas is dissolved in the liquid, so that

$$c(r,0) = \lim_{r\to\infty} c(r,t) = c_0. \tag{1.5}$$

Henry's law is assumed satisfied, so that

$$c(R+0,t) = kc_g(t), \tag{1.6}$$

where k is constant. It follows that, initially,

$$c_g(0) = c_0/k. \tag{1.7}$$

The previous paper showed that the hydrodynamic equations can be reduced to an ordinary differential equation relating the radius of the bubble to the total pressure within the bubble. We have

$$\rho(R\ddot{R}+3\dot{R}^2/2)+4\mu\dot{R}/R+2\sigma/R = Ac_g+p_rR_0^3/R^3-p_a, \tag{1.8}$$

in which p_a denotes the ambient pressure and ρ, μ, σ denote, respectively, the density, viscosity, and surface tension of the liquid. In order to obtain a determinate problem, the initial radius R_0 and the initial growth rate $\dot{R}_0$ should be specified.

In many cases of practical importance, the bubble-growth problem formulated above can be simplified, for certain of the physical phenomena involved may be completely negligible. By way of example, if the inertia of the liquid is negligible, equation (1.8) reduces to

$$4\mu\dot{R}/R+2\sigma/R = Ac_g+p_rR_0^3/R^3-p_a, \tag{1.9}$$

and there is no need to specify $\dot{R}_0$.

Whether or not a given phenomenon is negligible can properly be determined only when the bubble-growth problem is rewritten, in a meaningful way, in terms of dimensionless equations and boundary conditions. Such a dimensionless formulation also provides the similarity rules for scaling one bubble-growth problem to another.

Throughout this paper, it is assumed that the mathematical problem formulated above provides an adequate model of the physical problem. The limitations of the model were discussed in the previous paper.

2. Physical dimensions of the various parameters and variables

Let us note, first of all, that in the isothermal bubble-growth problem formulated in § 1 the only relevant basic physical dimensions are force F, length L, and time T. Force, rather than mass, is chosen as basic since an important special case is that of negligible inertia. There are two independent variables

$$[t] = T, \quad [r] = L. \tag{2.1}$$

The dependent variables of the problem, together with their dimensions, are given by

$$[p_g] = FL^{-2}, \quad [c_g] = FL^{-4}T^2, \quad [R] = L, \quad [v] = LT^{-1}, \quad [c] = FL^{-4}T^2. \tag{2.2}$$

The variables v and c depend upon both t and r; the others depend upon t alone.

In its full generality, the bubble-growth problem involves eleven parameters:

$$\left.\begin{array}{l}[R_0] = L, \quad [\dot{R}_0] = LT^{-1}, \quad [\rho] = FL^{-4}T^2, \quad [\sigma] = FL^{-1}, \\ [\mu] = FL^{-2}T, \quad [D] = L^2T^{-1}, \quad [c_0] = FL^{-4}T^2, \\ k, \text{ dimensionless}, \\ [A] = L^2T^{-2}, \quad [p_a] = FL^{-2}, \quad [p_r] = FL^{-2}.\end{array}\right\} \tag{2.3}$$

The selection of an appropriate length scale is quite straightforward: no matter which of the physical phenomena involved is negligible, R_0 is always an important parameter. Therefore, we let

$$r = R_0\lambda. \tag{2.4}$$

However, the meaningful selection of a time scale must be based upon some *a priori* knowledge of which physical effects will be dominant in a given problem. Otherwise, the scale chosen might involve as a factor some parameter which, in an important limiting case, tends to zero or which, in the limit, is not even relevant to the problem. When inertia is negligible, neither ρ nor $\dot{R}_0$ can be used: when surface tension is neglected, σ drops out of the problem; when the liquid can be taken as inviscid, μ cannot be used; when diffusion is unimportant, the parameters D, c_0, and k drop out automatically, and, furthermore, the problem can be reformulated so that A is not relevant; evidently, p_a and p_r, respectively, tend to zero in important limiting cases. Thus, there is no selection of a time scale which will remain meaningful in all limiting cases. In order to proceed, therefore, we must exclude from consideration, temporarily at least, some limiting cases. We shall assume that diffusion is important. Although this choice is somewhat arbitrary, it does correspond to the physical conditions assumed in the previous paper. A characteristic time of diffusion is the time required for the diffusion length $(DT)^{\frac{1}{2}}$ to equal the characteristic length scale of the problem. Thus, in keeping with our previous selection of R_0 as the length scale, we let

$$t = (R_0^2/D)\,\tau. \tag{2.5}$$

3. Viscosity-dominated hydrodynamics of growth

Let us consider first the bubble-growth problem under conditions such that the inertia and surface tension of the liquid can be completely neglected. In the next section we shall derive the criteria which determine when this is justified.

When inertia and surface tension are negligible, the bubble growth is governed by the ordinary differential equation

$$4\mu\, dR/dt = (Ac_g + p_r R_0^3/R^3 - p_a)\, R \tag{3.1}$$

and the partial differential equation

$$\frac{\partial c}{\partial t} + \frac{R^2}{r^2}\frac{dR}{dt}\frac{\partial c}{\partial r} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r}\right), \tag{3.2}$$

subject to the boundary conditions

$$(\partial c/\partial r)_{r=R} = (1/3DR^2)\, d(R^3 c_g)/dt, \tag{3.3}$$

$$\lim_{r\to\infty} c(r,t) = c_0, \tag{3.4}$$

to the initial condition

$$c(r,0) = c_0, \tag{3.5}$$

and to the matching condition

$$c(R+0,t) = kc_g(t). \tag{3.6}$$

We normalize each of the dependent variables with respect to its initial value

$$R = R_0\Lambda, \tag{3.7}$$

$$c = c_0 C, \tag{3.8}$$

$$c_g = (c_0/k)\,C_g. \tag{3.9}$$

With these normalized dependent variables and with the independent variables normalized according to equations (2.4) and (2.5), the bubble-growth problem becomes

$$\frac{1}{\Lambda}\frac{d\Lambda}{d\tau} = B\left(C_g + \frac{p_r}{p_0}\frac{1}{\Lambda_3} - \frac{p_a}{p_0}\right), \tag{3.10}$$

$$\frac{\partial C}{\partial \tau} + \frac{\Lambda^2}{\lambda^2}\frac{d\Lambda}{d\tau}\frac{\partial C}{\partial \lambda} = \frac{\partial^2 C}{\partial \lambda^2} + \frac{2}{\lambda}\frac{\partial C}{\partial \lambda}, \tag{3.11}$$

$$\frac{d}{d\tau}(\Lambda^3 C_g) = 3k\left(\frac{\partial C}{\partial \lambda}\right)_{\lambda=\Lambda}, \tag{3.12}$$

$$\lim_{\lambda\to\infty} C(\lambda,\tau) = 1, \tag{3.13}$$

$$C(\lambda,0) = 1, \tag{3.14}$$

$$C(\Lambda+0,\tau) = C_g(\tau), \tag{3.15}$$

in which p_0 is the partial pressure of solute gas initially within the bubble, i.e.

$$p_0 = Ac_0/k, \tag{3.16}$$

and the dimensionless parameter B (bubble number) is defined by

$$B = p_0 R_0^2/4\mu D = Ac_0 R_0^2/4k\mu D. \tag{3.17}$$

We see that, even without accounting for inertia and surface tension, the bubble-growth problem involves four dimensionless groupings: the pressure ratios p_r/p_0 and p_a/p_0, the parameter B, and the Henry's-law constant k. In order for two bubble-growth problems to be dynamically similar, each of these parameters, calculated for one problem, must have the same numerical value as when calculated for the other.

Let us now investigate the bubble-growth behaviour for extreme values of the parameters. The significance of the pressure ratios is clear: when

$$p_r/p_0 \ll 1, \tag{3.18}$$

the residual pressure is negligible compared with the partial pressure of the solute gas within the bubble; when

$$p_r/p_0 \gg 1, \tag{3.19}$$

the residual pressure dominates, at least during the early stages of growth. Ma & Wang (1962) have studied the effect of residual pressure on the dynamics of bubbles in inviscid liquids. When

$$p_a/p_0 \ll 1, \tag{3.20}$$

the presence of the ambient pressure can, in effect, be ignored, On the other hand, if p_a/p_0 approaches or exceeds unity, the bubble growth is severely retarded by the ambient pressure. Whenever

$$p_a/p_0 > C_g + (p_r/p_0)/\Lambda^3, \tag{3.21}$$

the bubble shrinks.

The parameter B is the ratio of R_0^2/D, the characteristic diffusion time, to $4\mu/p_0$, which is a characteristic time of initial growth, viz. it is the reciprocal of the logarithmic growth rate of a bubble under the driving pressure p_0, opposed by a liquid of viscosity μ and negligible surface tension.

When B is large, the bubble grows rapidly, compared with the typical speed at which solute gas diffuses toward it. For this case, the bubble-growth problem can be studied by using the *thin-shell approximation* employed by Plesset & Zwick (1954) in their study of thermal diffusion into a vapour bubble and applied to the problem of isothermal bubble growth by Barlow & Langlois (1962). In this approximation, the concentration of gas in the liquid is assumed to differ significantly from c_0 only in a thin shell surrounding the bubble.

On the other hand, when

$$B \ll 1, \tag{3.22}$$

the characteristic diffusion time is short compared with $4\mu/p_0$. The concentration just outside the bubble never differs appreciably from c_0 and, consequently, the partial pressure of solute gas in the bubble remains approximately equal to p_0. The bubble growth is then determined by the differential equation

$$(4\mu/R)\, dR/dt = p_0 - p_a + p_r R_0^3/R^3, \tag{3.23}$$

subject to the initial condition

$$R(0) = R_0. \tag{3.24}$$

Thus,

$$R/R_0 = \{\exp[3(p_0-p_a)t/4\mu] + p_r(\exp[3(p_0-p_a)t/4\mu]-1)/(p_0-p_a)\}^{\frac{1}{3}}. \tag{3.25}$$

The Henry's law constant k provides a measure of the solubility of the solute gas in the liquid. If k is zero, the gas is completely insoluble. Thus, the limiting case of k approaching zero with c_0 remaining finite is physically unrealistic, as evidenced by the breakdown of the normalization (3.9). However, it is meaningful to investigate the asymptotic behaviour of the bubble-growth equations when k and c_0 both approach zero in such a way that c_0/k, the initial concentration of gas in the bubble, remains finite. In this limiting case, the hydrodynamic problem of growth and the diffusion problem become uncoupled. The boundary condition (3.12) reduces to

$$d(\Lambda^3 C_g)/d\tau = 0. \tag{3.26}$$

Since, initially, C_g and Λ are both unity, integration of (3.26) yields

$$C_g = 1/\Lambda^3. \tag{3.27}$$

Integration of equation (3.10), subject to the initial condition

$$\Lambda(0) = 1, \tag{3.28}$$

then yields

$$\Lambda = \{[p_0 + p_r + (p_a - p_0 - p_r)\exp(-3BP_a\tau/p_0)]/p_a\}^{\frac{1}{3}}. \tag{3.29}$$

The diffusion problem can, in principle, now be solved separately: equation (3.11), with Λ given by equation (3.29), is integrated subject to the initial condition (3.14) and to the boundary conditions

$$\lim_{\lambda\to\infty} C(\lambda,\tau) = 1, \quad C(\Lambda+0,\tau) = p_a/[p_0+p_r+(p_a-p_0-p_r)\exp(-3Bp_a\tau/p_0)]. \tag{3.30}$$

When k is infinite, the vapour pressure of the solute gas is zero. Thus, if c_0 remains finite as k tends to infinity, the partial pressure of solute gas in the bubble tends to zero. Equation (3.1) then becomes

$$4\mu\, dR/dt = p_r R_0^3/R^2 - p_a R, \tag{3.31}$$

so that

$$R/R_0 = \{[p_r+(p_a-p_r)\exp(-3p_a t/4\mu)]/p_a\}^{\frac{1}{3}}. \tag{3.32}$$

Since no solute gas enters the bubble,

$$(\partial c/\partial r)_{r=R} = 0, \tag{3.33}$$

so that solution of the diffusion equation (3.2) subject to the boundary condition (3.4) and to the initial condition (3.5) yields

$$c(r,t) = c_0, \tag{3.34}$$

i.e., the gas concentration remains homogeneous.

The limiting case of k and c_0 both approaching infinity in such a way that their ratio approaches a finite limit is somewhat similar to the limiting case of B approaching zero. As k approaches infinity, the boundary condition (3.12) approaches

$$(\partial C/\partial \lambda)_{\lambda=\Lambda} = 0. \tag{3.35}$$

Hence, integration of (3.11) subject to (3.13) and (3.14) yields

$$C(\lambda,\tau) = 1. \tag{3.36}$$

The matching condition (3.15) then gives

$$C_g(\tau) = 1, \tag{3.37}$$

so that, as in the case of B approaching zero, the partial pressure of solute gas in the bubble remains at p_0. The bubble radius as a function of time is once more given by equation (3.25).

4. The effects of inertia and surface tension

When it cannot be guaranteed *a priori* that the inertia and surface tension of the liquid are negligible, the terms corresponding to these effects must be retained in the hydrodynamic equation (1·8). In terms of the normalized variables, defined by equations (2.4), (2.5), (3.7) and (3.9), equation (1.8) becomes

$$\frac{\rho D}{4\mu}\left[\Lambda\frac{d^2\Lambda}{d\tau^2}+\frac{3}{2}\left(\frac{d\Lambda}{d\tau}\right)^2\right]+\frac{1}{\Lambda}\frac{d\Lambda}{d\tau}+\frac{\sigma R_0}{2\mu D}\frac{1}{\Lambda} = B\left(C_g+\frac{p_r}{p_0}\frac{1}{\Lambda^3}-\frac{p_a}{p_0}\right). \tag{4.1}$$

This result would seem to indicate that inertia can be neglected in comparison with viscosity if and only if

$$\rho D/4\mu \ll 1, \tag{4.2}$$

and that surface tension can be neglected in comparison with viscosity if and only if

$$\sigma R_0/2\mu D \ll 1. \tag{4.3}$$

However, the quantities $d\Lambda/d\tau$ and $d^2\Lambda/d\tau^2$ are not necessarily of order unity. The dimensionless time τ was defined, according to equation (2.5), by normalizing with respect to a characteristic time of diffusion, not with respect to a characteristic time of hydrodynamic growth. As we saw in § 3, for extreme values of the parameter B these characteristic times are quite different. Thus, the criteria (4.2) and (4.3) are necessary and (with certain reservations to be discussed presently) sufficient for the neglect respectively, of inertia and surface tension only for moderate values of B.

For extreme values of B, it is necessary to specify a velocity v and an acceleration a which characterize the growth of the bubble. If this can be done, surface tension is negligible compared with viscosity only if

$$\sigma/2\mu v \ll 1. \tag{4.4}$$

There is a critical initial radius associated with surface tension. If the inertia terms in equation (1.8) are neglected, the growth rate of the bubble is zero when

$$R_0 = R_{\text{crit}} = 2\sigma/(p_0+p_r-p_a), \tag{4.5}$$

and when R_0 is smaller than R_{crit} the bubble shrinks. When R is nearly equal to R_{crit}, the characteristic growth velocity v is quite small, so that the obvious importance of surface tension in this case is reflected in the breakdown of the criterion (4.4). When (4.4) is satisfied, we normally expect that surface tension is negligible, However, this conclusion is not unequivocally assured, for it is always conceivable that a small term in a differential equation might, unexpectedly, produce a large effect. However, except for this reservation, which is always present in order of magnitude analyses, the criterion (4.4) is necessary and sufficient for the neglect of surface tension compared with viscosity.

Inertia is negligible compared with viscosity only if the criteria

$$3\rho R_0 v/8\mu \ll 1 \tag{4.6}$$

and

$$\rho R_0^2 a/4\mu v \ll 1 \tag{4.7}$$

are both satisfied. The criterion (4.6) could easily have been anticipated, for $3\rho R_0 v/8\mu$ is a Reynolds number. For the reason outlined in our discussion of the importance of surface tension, the criteria (4.6) and (4.7) are not quite sufficient to justify the neglect of inertia. In the case of inertia, there is available an evident example of their insufficiency. When it is known *a priori* that inertia is negligible, the initial growth rate is not specified. Instead, it is calculated by neglecting the inertia terms in equation (1.8):

$$v_0 = (dR/dt)_{t=0,\ \text{inertia neglected}} = (p_0+p_r-p_a-2\sigma/R_0)\,(R_0/4\mu). \tag{4.8}$$

However, when inertia may be important, $\dot{R}_0$ must be specified—and the specified value may be quite different from v_0, as defined by equation (4.8). If this is the case, the inertia of the liquid will be important in the first few instants of growth, even if the conditions (4.6) and (4.7) are satisfied. Thus, inertia produces noticeable transient effects unless

$$|(\dot{R}_0 - v_0)/v_0| \ll 1. \tag{4.9}$$

In summary, we have found that the isothermal bubble-growth problem is characterized by 8 dimensionless parameters:

(1) p_r/p_0, which measures the importance of the residual pressure of insoluble gases;

(2) p_a/p_0, which measures the importance of the ambient pressure;

(3) $B = Ac_0 R_0^2/4k\mu D$, which determines the relative magnitudes of the characteristic rates of gas diffusion and hydrodynamic growth;

(4) k the Henry's law constant, which determines the degree of coupling between the diffusion problem and the hydrodynamic problem;

(5) $\sigma/2\mu v$, which measures the relative importance of surface tension and viscosity;

(6) $3\rho R_0 v/8\mu$, the Reynolds number, which measures the importance of the inertia associated with the instantaneous rate of growth;

(7) $\rho R_0^2 a/4\mu v$, which measures the importance of the inertia associated with the accelerated rate of growth;

(8) $|(\dot{R}_0 - v_0)/v_0|$, which measures the transient effects of inertia.

REFERENCES

Barlow, E. J. & Langlois, W. E. 1962 Diffusion of gas from a liquid into an expanding bubble. *IBM J.* **6**, 329.

Ma, J. T. & Wang, P. K. 1962 Effect of initial air content on the dynamics of bubbles in liquids. *IBM J.* **6**, 472.

Plesset, M. S. & Zwick, S. A. 1954 The growth of vapor bubbles in superheated liquids. *J. Appl. Phys.* **25**, 393.

On the stable shape of subliming bodies in a high-enthalpy gas stream

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This paper describes a series of experiments carried out in a high-enthalpy stream of argon on materials that are known to sublime. The results confirm that an axisymmetric Teflon model ablates to a stable shape which is independent of the initial nose profile. The effect of changing the total enthalpy of the gas is simply to alter the recession rate of the nose. The experimental results show poor agreement with a simple theory which ignores the effects of mass transfer in the boundary layer.

1. Introduction

In the design of an ablation shield, consideration must be given to the change of shape of the shield during the re-entry phase. Since large changes in the profile will alter the aerodynamic characteristics of the vehicle, an effort must be made either to predict the change of shape or, better, to employ a nose profile that maintains its initial shape and recedes at a constant rate, namely the ablation velocity, V_W.

The problem of the equilibrium shape of any ablating material may be subdivided into: (*a*) when the stagnation temperature is approximately the same as the melting temperature of the body material; (*b*) when the stagnation temperature is much greater than the body melting (or sublimation) temperature. In the first case, melting occurs only in the stagnation region so that the nose of the body becomes more and more blunt until the heat-transfer rate is nearly uniform across the face. Examples of such behaviour have been given by Bogdonoff (1957), who describes a number of tests on ice models in a hypersonic stream of Mach number 13, stagnation pressure 100 lb. per square inch and a stagnation temperature of 294 °K. When the stagnation temperature is much greater than the body melting or vaporization temperature, experiments have indicated that after steady-state conditions are reached an equilibrium profile is attained which recedes at a constant rate. The explanation for this behaviour is that if the melting rate exceeds the equilibrium value, the temperature gradient at the surface is increased and more heat is conducted into the body, thus less heat is available for the melting process and the melt rate decreases. Conversely, if the melt rate falls below the equilibrium value the reverse procedure occurs.

McLellan (1955) described a series of experiments in which the melting of hemispherically-ended cylinders and cones was studied. The Woods-metal models were placed in an air stream at Mach number 6·9 with a stagnation temperature